Load balancing of dynamical distribution networks with flow constraints and unknown in/outflows

Wei, J.; Schaft, A.J. van der

Published in:
Systems & Control Letters

DOI:
10.1016/j.sysconle.2013.08.001

Publication date:
2013

Systems & Control Letters 62 (2013) 1001–1008

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Load balancing of dynamical distribution networks with flow constraints and unknown in/outflows

J. Wei *, A.J. van der Schaft

Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, PO Box 407, 9700 AK, The Netherlands

ARTICLE INFO

Article history:
Received 13 March 2013
Received in revised form
17 July 2013
Accepted 2 August 2013

Keywords:
PI controllers
Flow constraints
Directed graphs
Port-Hamiltonian systems
Consensus algorithms
Lyapunov stability

ABSTRACT

We consider a basic model of a dynamical distribution network, modeled as a directed graph with storage variables corresponding to every vertex and flow inputs corresponding to every edge, subject to unknown but constant inflows and outflows. As a preparatory result it is shown how a distributed proportional–integral controller structure, associating with every edge of the graph a controller state, will regulate the state variables of the vertices, irrespective of the unknown constant inflows and outflows, in the sense that the storage variables converge to the same value (load balancing or consensus). This will be proved by identifying the closed-loop system as a port-Hamiltonian system, and modifying the Hamiltonian function into a Lyapunov function, dependent on the value of the vector of constant inflows and outflows. In the main part of the paper the same problem will be addressed for the case that the input flow variables are *constrained* to take value in an arbitrary interval. We will derive sufficient and necessary conditions for load balancing, which only depend on the structure of the network in relation with the flow constraints.

1. Introduction

In this paper, we study a basic model for the dynamics of a distribution network. Identifying the network with a directed graph we associate with every vertex of the graph a state variable corresponding to *storage*, and with every edge a control input variable corresponding to *flow*, which is constrained to take value in a given closed interval. Furthermore, some of the vertices serve as terminals where an unknown but constant flow may enter or leave the network in such a way that the total sum of inflows and outflows is equal to zero. The control problem to be studied is to derive necessary and sufficient conditions for a distributed control structure (the control input corresponding to a given edge only depending on the difference of the state variables of the adjacent vertices) which will ensure that the state variables associated to all vertices will converge to the same value equal to the average of the initial condition, irrespective of the values of the constant unknown inflows and outflows.

The structure of the paper is as follows. Some preliminaries and notations will be given in Section 2. In Section 3 we will show how in the absence of constraints on the flow input variables a distributed proportional–integral (PI) controller structure, associating with every edge of the graph a controller state, will solve the problem if and only if the graph is weakly connected. This will be shown by identifying the closed-loop system as a port-Hamiltonian system, with state variables associated both to the vertices and the edges of the graph, in line with the general definition of port-Hamiltonian systems on graphs [1–4]; see also [5,6]. The proof of asymptotic load balancing will be given by modifying, depending on the vector of constant inflows and outflows, the total Hamiltonian function into a Lyapunov function. In the examples the obtained PI-controller often has a clear physical interpretation, emulating the physical action of adding energy storage and damping to the edges.

The main contribution of the paper resides in Sections 4 and 5, where the same problem is addressed for the case of *constraints* on the flow input variables. In Section 4 it will be shown that in the case of *zero* inflow and outflow the state variables of the vertices converge to the same value if and only if the network is strongly connected. This will be shown by constructing a C^1 Lyapunov function based on the total Hamiltonian and the constraint values. This same construction will be extended in Section 5 to the case of nonzero inflows and outflows, leading to the result that in this case asymptotic load balancing is reached if and only the graph is not only strongly connected but also *balanced*. Finally, Section 6 contains the conclusions.

Some preliminary results, in particular concerning Section 3, have been already reported before in [7].

* Corresponding author. Tel.: +31 0649125350.
E-mail addresses: jieqiang.wei@gmail.com, J.Wei@rug.nl (J. Wei), A.J.van.der.Schaft@rug.nl (A.J. van der Schaft).

0167-6911/$ – see front matter Crown Copyright © 2013 Published by Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.sysconle.2013.08.001

2. Preliminaries and notations

First we recall some standard definitions regarding directed graphs, as can be found e.g. in [8]. A *directed graph* $\mathcal{G}$ consists of a finite set $\mathcal{V}$ of *vertices* and a finite set $\mathcal{E}$ of *edges*, together with a mapping from $\mathcal{E}$ to the set of ordered pairs of $\mathcal{V}$, where no self-loops are allowed. Thus to any edge $e \in \mathcal{E}$ there corresponds an ordered pair $(v, w) \in \mathcal{V} \times \mathcal{V}$ (with $v \neq w$), representing the tail vertex v and the head vertex w of this edge.

A directed graph is completely specified by its *incidence matrix* B, which is an $n \times m$ matrix, n being the number of vertices and m being the number of edges, with (i, j)th element equal to 1 if the jth edge is towards vertex i, and equal to -1 if the jth edge is originating from vertex i, and 0 otherwise. Since we will only consider the directed graphs in this paper 'graph' will throughout mean 'directed graph' in the sequel. A directed graph is *strongly connected* if it is possible to reach any vertex starting from any other vertex by traversing edges following their directions. A directed graph is called *weakly connected* if it is possible to reach any vertex from every other vertex using the edges *not* taking into account their direction. A graph is weakly connected if and only if $\ker B^T = \operatorname{span} \mathbb{1}_n$. Here $\mathbb{1}_n$ denotes the n-dimensional vector with all elements equal to 1. A graph that is not weakly connected falls apart into a number of weakly connected subgraphs, called the connected components. The number of connected components is equal to $\dim \ker B^T$. For each vertex, the number of incoming edges is called the *in-degree* of the vertex and the number of outgoing edges its *out-degree*. A graph is called *balanced* if for every vertex their in-degree and out-degree of every vertex are equal. A graph is balanced if and only if $\mathbb{1}_n \in \ker B$.

Given a graph, we define its *vertex space* as the vector space of all functions from $\mathcal{V}$ to some linear space $\mathcal{R}$. In the rest of this paper we will take for simplicity $\mathcal{R} = \mathbb{R}$, in which case the vertex space can be identified with $\mathbb{R}^n$. Similarly, we define its *edge space* as the vector space of all functions from $\mathcal{E}$ to $\mathcal{R} = \mathbb{R}$, which can be identified with $\mathbb{R}^m$. In this way, the incidence matrix B of the graph can be also regarded as the matrix representation of a linear map from the edge space $\mathbb{R}^m$ to the vertex space $\mathbb{R}^n$.

Notation: for $a, b \in \mathbb{R}^m$ the notation $a \leqslant b$ will denote elementwise inequality $a_i \leq b_i$, $i = 1, \dots, m$. For $a_i < b_i$, $i = 1, \dots, m$ the multidimensional saturation function $\operatorname{sat}(x; a, b) : \mathbb{R}^m \to \mathbb{R}^m$ is defined as

$$\operatorname{sat}(x; a, b)_i = \begin{cases} a_i & \text{if } x_i \leq a_i, \\ x_i & \text{if } a_i < x_i < b_i, \\ b_i & \text{if } x_i \geq b_i, \end{cases} \quad i = 1, \dots, m. \tag{1}$$

3. A dynamic network model with PI controller

Let us consider the following dynamical system defined on the graph $\mathcal{G}$

$$\begin{aligned} \dot{x} &= Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\ y &= B^T \frac{\partial H}{\partial x}(x), \quad y \in \mathbb{R}^m, \end{aligned} \tag{2}$$

where $H : \mathbb{R}^n \to \mathbb{R}$ is any differentiable function, and $\frac{\partial H}{\partial x}(x)$ denotes the column vector of partial derivatives of H. Here the ith element x_i of the state vector x is the state variable associated to the ith vertex, while u_j is a flow input variable associated to the jth edge of the graph. System (2) defines a port-Hamiltonian system [9,10], satisfying the energy-balance

$$\frac{\mathrm{d}}{\mathrm{d}t} H = u^T y. \tag{3}$$

Note that geometrically its state space is the vertex space, its input space is the edge space, while its output space is the dual of the edge space.

Example 3.1 (*Hydraulic Network*)**.** Consider a hydraulic network, modeled as a directed graph with vertices (nodes) corresponding to reservoirs, and edges (branches) corresponding to pipes. Let x_i be the amount of water stored at vertex i, and u_j the flow through edge j. Then the mass-balance of the network is summarized in

$$\dot{x} = Bu, \tag{4}$$

where B is the incidence matrix of the graph. Let furthermore $H(x)$ denote the stored energy in the reservoirs (e.g., gravitational energy). Then $P_i := \frac{\partial H}{\partial x_i}(x)$, $i = 1, \dots, n$, are the *pressures* at the vertices, and the output vector $y = B^T \frac{\partial H}{\partial x}(x)$ is the vector whose jth element is the pressure *difference* $P_i - P_k$ across the jth edge linking vertex k to vertex i.

As a next step we will extend the dynamical system (2) with a vector d of *inflows and outflows*

$$\begin{aligned} \dot{x} &= Bu + Ed, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, \ d \in \mathbb{R}^k \\ y &= B^T \frac{\partial H}{\partial x}(x), \quad y \in \mathbb{R}^m, \end{aligned} \tag{5}$$

with E an $n \times k$ matrix whose columns consist of exactly one entry equal to 1 (inflow) or -1 (outflow), while the rest of the elements is zero. Thus E specifies the k terminal vertices where flows can enter or leave the network.

In this paper we will regard d as a vector of constant *disturbances*, and we want to investigate control schemes which ensure asymptotic load balancing of the state vector x irrespective of the (unknown) disturbance d. The simplest control possibility is to apply a proportional output feedback

$$u = -Ry = -RB^T \frac{\partial H}{\partial x}(x), \tag{6}$$

where R is a diagonal matrix with strictly positive diagonal elements $r_1, \dots, r_m$. Note that this defines a *decentralized* control scheme if H is of the form $H(x) = H_1(x_1) + \cdots + H_n(x_n)$, in which case the ith input as given by (6) equals r_i times the difference of the component of $\frac{\partial H}{\partial x}(x)$ corresponding to the head vertex of the ith edge and the component of $\frac{\partial H}{\partial x}(x)$ corresponding to its tail vertex. This control scheme leads to the closed-loop system

$$\dot{x} = -BRB^T \frac{\partial H}{\partial x}(x) + Ed. \tag{7}$$

In the case of *zero* in/outflows $d = 0$ this implies the energy-balance

$$\frac{\mathrm{d}}{\mathrm{d}t} H = -\frac{\partial^T H}{\partial x}(x) BRB^T \frac{\partial H}{\partial x}(x) \leq 0. \tag{8}$$

Hence if H is radially unbounded it follows that the system trajectories of the closed-loop system (7) will converge to the set

$$\mathcal{E} := \left\{ x \mid B^T \frac{\partial H}{\partial x}(x) = 0 \right\} \tag{9}$$

and thus to the load balancing set

$$\mathcal{E} = \left\{ x \mid \frac{\partial H}{\partial x}(x) = \alpha \mathbb{1}, \alpha \in \mathbb{R} \right\}$$

if and only if $\ker B^T = \operatorname{span}\{\mathbb{1}\}$, or equivalently [8], if and only if the graph is *weakly connected*.

In particular, for the standard Hamiltonian $H(x) = \frac{1}{2}\|x\|^2$ this means that the state variables $x_i(t)$, $i = 1, \dots, n$, converge to a common value α as $t \to \infty$. Since $\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{1}^T x(t) = 0$ it follows that this common value is given as $\alpha = \frac{1}{n} \sum_{i=1}^n x_i(0)$.

For $d \neq 0$ proportional control $u = -Ry$ will not be sufficient to reach load balancing, since the disturbance d can only be attenuated at the expense of increasing the gains in the matrix R. Hence we consider *proportional–integral* (PI) control given by the dynamic output feedback[1]

$$\dot{x}_c = y,$$
$$u = -Ry - \frac{\partial H_c}{\partial x_c}(x_c), \tag{10}$$

where $H_c(x_c)$ denotes the storage function (energy) of the controller. Note that, this PI controller is of the same decentralized nature as the static output feedback $u = -Ry$.

The jth element of the controller state x_c can be regarded as an additional state variable corresponding to the jth edge. Thus $x_c \in \mathbb{R}^m$, the edge space of the network. The closed-loop system resulting from the PI control (10) is given as

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} -BRB^T & -B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x}(x) \\ \frac{\partial H_c}{\partial x_c}(x_c) \end{bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix} d, \tag{11}$$

This is again a port-Hamiltonian system,[2] with total Hamiltonian $H_{\text{tot}}(x, x_c) := H(x) + H_c(x_c)$, and satisfying the energy-balance

$$\frac{\mathrm{d}}{\mathrm{d}t} H_{\text{tot}} = -\frac{\partial^T H}{\partial x}(x) BRB^T \frac{\partial H}{\partial x}(x) + \frac{\partial^T H}{\partial x}(x) E d \tag{12}$$

Consider now a constant disturbance $\bar{d}$ for which there exists a *matching* controller state $\bar{x}_c$, i.e.,

$$E\bar{d} = B \frac{\partial H_c}{\partial x_c}(\bar{x}_c). \tag{13}$$

This allows us to modify the total Hamiltonian $H_{\text{tot}}(x, x_c)$ into[3]

$$V_{\bar{d}}(x, x_c) := H(x) + H_c(x_c) - \frac{\partial^T H_c}{\partial x_c}(\bar{x}_c)(x_c - \bar{x}_c) - H_c(\bar{x}_c), \tag{14}$$

which will serve as a candidate Lyapunov function; leading to the following theorem.

Theorem 1. *Consider the system (5) on the graph $\mathcal{G}$ in closed loop with the PI-controller (10). Let the constant disturbance $\bar{d}$ be such that there exists a $\bar{x}_c$ satisfying the matching equation (13). Assume that $V_{\bar{d}}(x, x_c)$ is radially unbounded. Then the trajectories of the closed-loop system (11) will converge to an element of the load balancing set*

$$\mathcal{E}_{\text{tot}} = \left\{ (x, x_c) \mid \frac{\partial H}{\partial x}(x) = \alpha \mathbb{1},\ \alpha \in \mathbb{R}, B \frac{\partial H_c}{\partial x_c}(x_c) = E\bar{d} \right\} \tag{15}$$

if and only if $\mathcal{G}$ is weakly connected.

Proof. Suppose that $\mathcal{G}$ is weakly connected. By (12) for $d = \bar{d}$ we obtain, making use of (13),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V_{\bar{d}} &= -\frac{\partial^T H}{\partial x}(x) BRB^T \frac{\partial H}{\partial x}(x) + \frac{\partial^T H}{\partial x}(x) E\bar{d} \\ &\quad - \frac{\partial^T H_c}{\partial x_c}(\bar{x}_c) B^T \frac{\partial H}{\partial x}(x) \\ &= -\frac{\partial^T H}{\partial x}(x) BRB^T \frac{\partial H}{\partial x}(x) \leq 0. \end{aligned} \tag{16}$$

Hence by LaSalle's invariance principle the system trajectories converge to the largest invariant set contained in

$$\left\{ (x, x_c) \mid B^T \frac{\partial H}{\partial x}(x) = 0 \right\}.$$

Substitution of $B^T \frac{\partial H}{\partial x}(x) = 0$ in the closed-loop system equations (11) yields x_c constant and $-B \frac{\partial H_c}{\partial x_c}(x_c) + E\bar{d} = 0$. Since the graph is weakly connected $B^T \frac{\partial H}{\partial x}(x) = 0$ implies $\frac{\partial H}{\partial x}(x) = \alpha \mathbb{1}$. If the graph is not weakly connected then the above analysis will hold on every connected component, and the common value α will be different for different components. □

Corollary 2. *If* $\ker B = 0$*, which is equivalent [8] to the graph having no cycles, then for every $\bar{d}$ there exists a unique $\bar{x}_c$ satisfying (13), and convergence is towards the set $\mathcal{E}_{\text{tot}} = \left\{ (x, \bar{x}_c) \mid \frac{\partial H}{\partial x}(x) = \alpha \mathbb{1}, \alpha \in \mathbb{R},\ x_c = \bar{x}_c \right\}$.*

Corollary 3. *In the case of the standard quadratic Hamiltonians $H(x) = \frac{1}{2}\|x\|^2$, $H_c(x_c) = \frac{1}{2}\|x_c\|^2$ there exists for every $\bar{d}$ a controller state $\bar{x}_c$ such that (13) holds if and only if*

$$\operatorname{im} E \subset \operatorname{im} B. \tag{17}$$

Furthermore, in this case $V_{\bar{d}}$ equals the radially unbounded function $\frac{1}{2}\|x\|^2 + \frac{1}{2}\|x_c - \bar{x}_c\|^2$, while convergence will be towards the load balancing set $\mathcal{E}_{\text{tot}} = \{(x, x_c) \mid x = \alpha \mathbb{1}, \alpha \in \mathbb{R},\ Bx_c = E\bar{d}\}$.

A necessary (and in the case the graph is weakly connected necessary *and* sufficient) condition for the inclusion $\operatorname{im} E \subset \operatorname{im} B$ is that $\mathbb{1}^T E = 0$. In its turn $\mathbb{1}^T E = 0$ is equivalent to the fact that for every $\bar{d}$ the total inflow into the network equals to the total outflow. The condition $\mathbb{1}^T E = 0$ also implies

$$\mathbb{1}^T \dot{x} = -\mathbb{1}^T BRB^T \frac{\partial H}{\partial x}(x) + \mathbb{1}^T E\bar{d} = 0, \tag{18}$$

yielding (as in the case $d = 0$) that $\mathbb{1}^T x$ is a *conserved quantity* for the closed-loop system (11). In particular it follows that the limit value $\lim_{t\to\infty} x(t) \in \operatorname{span}\{\mathbb{1}\}$ is determined by the initial condition $x(0)$.

Example 3.2 (*Hydraulic Network Continued*)**.** The proportional part $u = -Ry$ of the controller corresponds to adding *damping* to the dynamics (proportional to the pressure differences along the edges). The integral part of the controller has the interpretation of adding *compressibility* to the hydraulic network dynamics. Using this emulated compressibility, the PI-controller is able to regulate the hydraulic network to a load balancing situation where all pressures P_i are equal, irrespective of the constant inflow and outflow $\bar{d}$ satisfying the matching condition (13). Note that for the Hamiltonian $H(x) = \frac{1}{2}\|x\|^2$ the pressures P_i are equal to each other if and only if the water levels x_i are equal.

4. Constrained flows: the case without in/outflows

In many cases of interest, the elements of the vector of flow inputs $u \in \mathbb{R}^m$ corresponding to the edges of the graph will be *constrained*, that is

$$u \in \mathcal{U} := \{u \in \mathbb{R}^m \mid u^- \leqslant u \leqslant u^+\} \tag{19}$$

for certain vectors u^- and u^+ satisfying $u_i^- \leqslant 0 \leqslant u_i^+$, $i = 1, \ldots, m$ (throughout $\leqslant$ denotes element-wise inequality). This leads to the

[1] The same strategy and analysis for handling constant disturbances in port-Hamiltonian systems was already given in [11].

[2] See [1–4] for a general definition of port-Hamiltonian systems on graphs. The addition of a PI-controller can be also interpreted as 'control by interconnection', see e.g. [12].

[3] This function was introduced for passive systems with constant inputs in [13].

following constrained version[4] of the PI controller (10) given in the previous section

$$\dot{x}_c = y,$$
$$u = \mathrm{sat}\left(-Ry - \frac{\partial H_c}{\partial x_c}(x_c); u^-, u^+\right). \tag{20}$$

Throughout this paper we make the following assumption on the flow constraints.

Assumption 4.

$$u_i^- \leqslant 0, \qquad u_i^+ \geqslant 0, \qquad u_i^- < u_i^+, \quad i = 1, \ldots, m. \tag{21}$$

It is important to note that we may change the *orientation* of some of the edges of the graph at will; replacing the corresponding columns b_i of the incidence matrix B by $-b_i$. Noting the identity $\mathrm{sat}(-x; u_i^-, u_i^+) = -\mathrm{sat}(x; -u_i^+, -u_i^-)$ this implies that we may assume without loss of generality that the orientation of the graph is chosen such that

$$u_i^- \leqslant 0 < u_i^+, \quad i = 1, \ldots, m. \tag{22}$$

This will be assumed throughout the rest of the paper. In general, we will say that any orientation of the graph is *compatible* with the flow constraints if (22) holds. If the j-th edge is such that $u_j^- = 0$ then we will call this edge an *uni-directional* edge, while if $u_j^- < 0$ then the edge is called a *bi-directional* edge.

In this section, we will first analyze the closed-loop system for the constrained PI-controller under the simplifying assumption of *zero inflow and outflow* ($d = 0$). In the next section, we will deal with the general case. Furthermore, for the simplicity of exposition we consider throughout the rest of this paper the standard Hamiltonian $H_c(x_c) = \frac{1}{2}\|x_c\|^2$ for the constrained PI controller and the identity gain matrix $R = I$, while we also throughout assume that the Hessian matrix of Hamiltonian $H(x)$ is positive definite for any x. Thus we consider in the rest of this section the closed-loop system

$$\dot{x} = B\mathrm{sat}\left(-B^T \frac{\partial H}{\partial x}(x) - x_c;\; u^-, u^+\right),$$
$$\dot{x}_c = B^T \frac{\partial H}{\partial x}(x). \tag{23}$$

In order to state the main theorem of this section we need one more definition.

Definition 5. Consider the directed graph $\mathcal{G}$ together with the constraint values u^-, u^+ satisfying (22). Then we will call the graph strongly connected *with respect to the flow constraints* $u^- \leqslant u \leqslant u^+$ if the following holds: for every two vertices v_1, v_2 there exists an orientation of the graph compatible with the flow constraints[5] and a directed path (directed with respect to this orientation) from v_1 to v_2.

Theorem 6. *Consider the closed-loop system* (23) *on a graph* $\mathcal{G}$ *with flow constraints* $u^- \leqslant u \leqslant u^+$ *satisfying* (22). *Then its solutions converge to the load balancing set*

$$\mathcal{E}_{\mathrm{tot}} = \left\{(x, x_c) \mid \frac{\partial H}{\partial x}(x) = \alpha \mathbb{1}_n,\ B\mathrm{sat}(-x_c; u^-, u^+) = 0\right\} \tag{24}$$

if and only if the graph is strongly connected with respect to the flow constraints.

Proof. *Sufficiency*: consider the Lyapunov function given by

$$V(x, x_c) = \mathbb{1}_m^T S\left(-B^T \frac{\partial H}{\partial x}(x) - x_c; u^-, u^+\right) + H(x), \tag{25}$$

with

$$S(x; u^-, u^+)_i := \int_0^{x_i} \mathrm{sat}(y; u_i^-, u_i^+)\, dy. \tag{26}$$

It can be easily verified that V is positive definitive, radially unbounded and C^1. Its time-derivative is given as

$$\begin{aligned}\dot{V} &= -\mathrm{sat}^T\left(-B^T \frac{\partial H}{\partial x}(x) - x_c; u^-, u^+\right) B^T \\ &\quad \times B\mathrm{sat}\left(-B^T \frac{\partial H}{\partial x}(x) - x_c; u^-, u^+\right) \\ &\leqslant 0.\end{aligned} \tag{27}$$

By LaSalle's invariance principle, all trajectories will converge to the largest invariant set, denoted as $\mathcal{I}$, contained in $\mathcal{K} = \left\{(x, x_c) \mid B\ \mathrm{sat}\left(-B^T \frac{\partial H}{\partial x}(x) - x_c; u^-, u^+\right) = 0\right\}$. Whenever $x \in \mathcal{K}$ it follows that $\dot{x} = 0$ and thus $x(t) = \nu$ for some constant vector ν. Hence, since $\dot{x}_c = B^T \frac{\partial H}{\partial x}(x)$, it follows that $x_c(t) = B^T \frac{\partial H}{\partial x}(\nu)t + x_c(0)$.

Suppose now that $B^T \frac{\partial H}{\partial x}(\nu) \neq 0$. Then for t large enough

$$\begin{aligned}0 &= \frac{\partial^T H}{\partial x}(\nu) B\mathrm{sat}\left(-B^T \frac{\partial H}{\partial x}(\nu) - B^T \frac{\partial H}{\partial x}(\nu)t - x_c(0); u^-, u^+\right) \\ &= \sum_{i=1}^m \left(B^T \frac{\partial H}{\partial x}(\nu)\right)_i c_i,\end{aligned} \tag{28}$$

where

$$c_i = \begin{cases} u_i^- & \text{if } \left(B^T \frac{\partial H}{\partial x}(\nu)\right)_i > 0, \\ u_i^+ & \text{if } \left(B^T \frac{\partial H}{\partial x}(\nu)\right)_i < 0. \end{cases} \tag{29}$$

Hence, in view of $u_i^- \leqslant 0 < u_i^+$, we have $\left(B^T \frac{\partial H}{\partial x}(\nu)\right)_i \geqslant 0$, for $i = 1, \ldots, m$. However since the graph is strongly connected with respect to the flow constraints, if $\left(B^T \frac{\partial H}{\partial x}(\nu)\right)_i > 0$, then there exists j such that $\left(B^T \frac{\partial H}{\partial x}(\nu)\right)_j < 0$. This yields a contradiction. We conclude that $B^T \frac{\partial H}{\partial x}(\nu) = 0$, which implies $\frac{\partial H}{\partial x}(\nu) = \alpha \mathbb{1}_n$, and thus all trajectories converge to $\mathcal{E}_{\mathrm{tot}}$.

Necessity: assume without loss of generality that the graph is weakly connected. (Otherwise the same analysis can be performed on every connected component.) If the graph is not strongly connected with respect to the flow constraints then there is a pair of vertices v_i, v_j for which there exist a compatible orientation and a directed path from v_i to v_j, but not a compatible orientation and directed path from v_j to v_i. In other words, there can be positive flow from v_i to v_j, but not vice versa. Then for suitable initial condition, $\frac{\partial H}{\partial x_i}x(t) < \frac{\partial H}{\partial x_j}x(t)$ for all $t \geqslant 0$, and thus there is no convergence to $\mathcal{E}_{\mathrm{tot}}$. $\square$

Remark 7. Note that for $u_i^- \to -\infty$, $u_i^+ \to \infty$, the Lyapunov function (25) tends to the function $H(x) + \frac{1}{2}\|B^T \frac{\partial H}{\partial x}(x) + x_c\|^2$, which is different from the Lyapunov function $H(x) + \frac{1}{2}\|x_c\|^2$ used in the previous section.

In the special case that the flow constraints are such that *all* the flows u_i can follow both directions, we obtain the following corollary.

[4] See also [14,15] for a related problem setting where a constrained version of the proportional controller (6) is considered.

[5] Note that for different pairs of vertices we may need different orientations compatible with the flow constraints. Thus the definition of strong connectedness with respect to the flow constraints is *weaker* than the existence of an orientation of the graph compatible with the flow constraints in which the graph is strongly connected.

Corollary 8. *For a network with constraint intervals* $[u_i^-, u_i^+]$ *with* $u_i^- < 0 < u_i^+, i = 1, \ldots, m$, *the trajectories of the closed-loop system* (23) *will converge to the set* (24) *if and only if the network is weakly connected.*

Proof. In this case (since all the edges are bi-directional) the weak connectedness is equivalent to the strong connectedness with respect to the flow constraints. If the graph is not weakly connected then the components of $\frac{\partial H}{\partial x}(x)$ will only converge to a common value on every connected component. □

5. Nonzero inflows and outflows

In this section, we deal with the general case of nonzero (but constant) inflows and outflows $\bar{d}$. Thus we consider the closed-loop system

$$\begin{aligned} \dot{x} &= B\mathrm{sat}\left(-B^T \frac{\partial H}{\partial x}(x) - x_c; u^-, u^+\right) + E\bar{d}, \\ \dot{x}_c &= B^T \frac{\partial H}{\partial x}(x), \end{aligned} \tag{30}$$

with $\mathrm{im}E \subset \mathrm{im}B$.

In order for the system to reach consensus, we need to impose conditions on the magnitude of the in/outflows $\bar{d}$.

Definition 9. Given the constraint values $u^- < u^+$ the *permission set* $\mathcal{P}(u^-, u^+)$ is defined as

$$\mathcal{P}_1(u^-, u^+) \times \mathcal{P}_2(u^-, u^+) \cdots \times \mathcal{P}_m(u^-, u^+)$$

where the intervals $\mathcal{P}_i(u^-, u^+)$ are defined by:

$$\mathcal{P}_i(u^-, u^+) = \begin{cases} (u_i^-, -u_i^-) & \text{if } 0 \in (u_i^-, u_i^+) \text{ and } |u_i^-| \leqslant |u_i^+| \\ (-u_i^+, u_i^+) & \text{if } 0 \in (u_i^-, u_i^+) \text{ and } |u_i^-| > |u_i^+| \\ (0, u_{\min}^+) & \text{if } (u_i^-, u_i^+) = (0, u_i^+), \end{cases} \tag{31}$$

where $u_{\min}^+ = \min\{u_i^+ \mid i \text{ such that } u_i^- = 0\}$.

Definition 10 (*Matching Condition for the Constrained Case*)**.** Given the constraint vectors u^- and u^+, the in/outflows will be said to satisfy the matching condition for the constrained case if there exists an $\bar{x}_c \in \mathcal{P}(u^-, u^+)$ such that $E\bar{d} = B\bar{x}_c$.

Theorem 11. *Consider a graph* $\mathcal{G}$ *with dynamics* (30)*. Suppose that every edge allows bi-directional flow, i.e.,* $u_i^- < 0 < u_i^+, i = 1, \ldots, m$*. Then for any in/outflow* $\bar{d}$ *satisfying the matching condition, the trajectories of* (30) *will converge to*

$$\mathcal{E}_{\mathrm{tot}} = \left\{ (x, x_c) \mid \frac{\partial H}{\partial x}(x) = \alpha \mathbb{1}, \ \alpha \in \mathbb{R}, \quad B\mathrm{sat}(-x_c; u^-, u^+) + E\bar{d} = 0 \right\} \tag{32}$$

if and only if the graph $\mathcal{G}$ *is weakly connected.*

Proof. By the matching condition $E\bar{d} = B\bar{x}_c$ and the identity

$$\mathrm{sat}(x - \eta; u^-, u^+) + \eta = \mathrm{sat}(x; u^- + \eta, u^+ + \eta), \quad \forall \eta \in \mathbb{R}^n, \tag{33}$$

the system (30) can be written as

$$\begin{aligned} \dot{x} &= B\mathrm{sat}\left(-B^T \frac{\partial H}{\partial x}(x) - \tilde{x}_c; u^- + \bar{x}_c, u^+ + \bar{x}_c\right), \\ \dot{\tilde{x}}_c &= B^T \frac{\partial H}{\partial x}(x), \end{aligned} \tag{34}$$

where $\tilde{x}_c = x_c - \bar{x}_c$. Since by construction $(u^- + \bar{x}_c)_i < 0$, $i = 1, \ldots, m$, the proof now follows from Corollary 8. □

The following theorem covers the case that every edge is uni-directional.

Theorem 12. *Consider a network* $\mathcal{G}$ *with dynamics* (30) *with flow constraints such that* $u_i^- = 0, i = 1, \ldots, m$ *(uni-directional flow). Then for any* $u^+ \in \mathbb{R}_+^m$ *and any in/outflow* $\bar{d}$ *satisfying the matching condition for the constrained case, the trajectories of* (30) *converge to*

$$\mathcal{E}_{\mathrm{tot}} = \left\{ (x, x_c) \mid \frac{\partial H}{\partial x}(x) = \alpha \mathbb{1}, \ \alpha \in \mathbb{R}, \quad B\mathrm{sat}(-x_c; 0_m, u^+) + E\bar{d} = 0 \right\} \tag{35}$$

if and only if the graph in the (only) orientation compatible with the flow constraints is strongly connected and balanced.

In order to prove Theorem 12 we need the following two lemmas. Recall that a directed graph is *balanced* if every vertex has in-degree (number of incoming edges) equal to out-degree (number of outgoing edges). Furthermore, we will say that two *cycles* of a graph are *non-overlapping* if they do not have any edges in common.

Lemma 13. *A strongly connected graph is balanced if and only if it can be covered by non-overlapping cycles.*

Proof. *Sufficiency*: if a graph can be covered by non-overlapping cycles, then every vertex necessarily has the same in-degree and out-degree; so this graph is balanced.

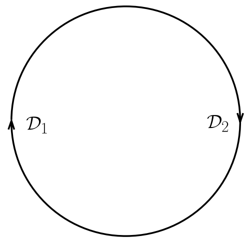
Necessity: since the graph is strongly connected, every two vertices can be connected by a directed path, and the graph can be covered by cycles. Now suppose that the graph can*not* be covered by non-overlapping cycles. We will show that this implies that the graph is not balanced.

Let k be the smallest number of cycles needed to cover the graph, and let $\mathcal{T} = (C_1, C_2, \ldots, C_k)$ be a covering set of cycles. According to our assumption, at least one edge of the graph is shared by two or more cycles in $\mathcal{T}$. We claim that the set of shared edges cannot contain any cycles. Indeed, suppose that there is one cycle, denoted as $\mathcal{D}$ (depicted in Fig. 1(a)), whose edges are all shared by elements of $\mathcal{T}$. If $\mathcal{D} \in \mathcal{T}$, then obviously $\mathcal{T}$ is not a minimal covering set, since by deleting the cycle $\mathcal{D}$ from $\mathcal{T}$ we have a covering set of $k - 1$ elements.

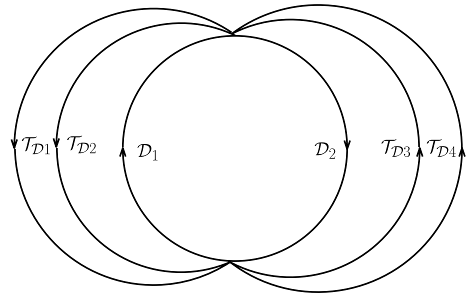
Thus $\mathcal{D} \notin \mathcal{T}$. It can be seen that the minimal number c of cycles in $\mathcal{T}$ which cover $\mathcal{D}$ twice is at least 4. Denote such a minimal set of c cycles in $\mathcal{T}$ which cover $\mathcal{D}$ by $\mathcal{T}_{\mathcal{D}}$. We will now show that by combining these c cycles with the cycle $\mathcal{D}$ there exist 3 cycles in the original graph $\mathcal{G}$ which cover the subgraph given by $\mathcal{T}_{\mathcal{D}}$; thus reaching a contradiction with the minimality of $\mathcal{T}$. The construction of these 3 cycles is indicated in Fig. 1. Consider for simplicity the case that 4 cycles in $\mathcal{T}$, denoted by C_1, C_2, C_3, C_4 cover $\mathcal{D}$ twice. Combining (depending on the orientation of the cycles) part of C_1 with part of C_3, and part of C_2 with part of C_4 (see Fig. 1), we can define 2 cycles which together with the cycle $\mathcal{D}$ yield a set of 3 cycles which cover the subgraph spanned by C_1, C_2, C_3, C_4.

In conclusion, there must exist at least one shared edge, say (v_i, v_j), such that all edges with tail-vertex v_j are used only once in $\mathcal{T}$. But this implies that v_j has larger out-degree than in-degree, i.e., the graph is not balanced. □

Lemma 14. *Consider a strongly connected and balanced graph with dynamics* (30) *with flow constraints and disturbance as given in Theorem* 12*. Then the following statements hold:*



(a) The cycle $\mathcal{D}$ split into two parts $\mathcal{D}_1$, $\mathcal{D}_2$.



(b) The subgraph $\mathcal{T}_{\mathcal{D}}$.

Fig. 1. (a) The cycle $\mathcal{D}$. We divide the edges of $\mathcal{D}$ into two disjoint sets: $\mathcal{D}_1$ contains the left part of $\mathcal{D}$ and $\mathcal{D}_2$ contains the rest. (b) The subgraph given by $\mathcal{T}_{\mathcal{D}}$. Without $\mathcal{D}$, we need at least 4 cycles to cover $\mathcal{D}$ twice; these cycles are given as $C_1 = \mathcal{T}_{\mathcal{D}1} \cup \mathcal{D}_1$, $C_2 = \mathcal{T}_{\mathcal{D}2} \cup \mathcal{D}_1$, $C_3 = \mathcal{T}_{\mathcal{D}3} \cup \mathcal{D}_2$ and $C_4 = \mathcal{T}_{\mathcal{D}4} \cup \mathcal{D}_2$. It follows that $\mathcal{T}_{\mathcal{D}}$ is also covered by the 3 cycles: $\mathcal{D}$(clockwise), $\mathcal{T}_{\mathcal{D}2} \cup \mathcal{T}_{\mathcal{D}3}$ (counterclockwise) and $\mathcal{T}_{\mathcal{D}1} \cup \mathcal{T}_{\mathcal{D}4}$ (counterclockwise).

(i) *along every trajectory $(x(t), x_c(t))$, $t \geqslant 0$, of (30), the function*

$$V(x, \tilde{x}_c) = \mathbb{1}^T S\left(-B^T \frac{\partial H}{\partial x}(x) - \tilde{x}_c; \bar{x}_c, u^+ + \bar{x}_c\right) + H(x) \quad (36)$$

is bounded from below,

(ii) *the trajectory $(x(t), \tilde{x}_c(t))$, $t \geqslant 0$, is bounded,*

(iii) $\lim_{t\to\infty} \dot{V}(x(t), \tilde{x}_c(t)) = 0$,

where $\tilde{x}_c = x_c - \bar{x}_c$.

Proof. (i) By using (33) and the matching condition for the constrained case, we rewrite the system as

$$\begin{aligned}\dot{x} &= B\mathrm{sat}\left(-B^T \frac{\partial H}{\partial x}(x) - \tilde{x}_c; \bar{x}_c, u^+ + \bar{x}_c\right),\\ \dot{\tilde{x}}_c &= B^T \frac{\partial H}{\partial x}(x),\end{aligned} \quad (37)$$

where $\tilde{x}_c = x_c - \bar{x}_c$. Since for a balanced network $B\mathbb{1} = 0$, we obtain the following implications

$$\begin{aligned}0 = {}& \mathbb{1}^T B^T \frac{\partial H}{\partial x}(x)\\ &\Rightarrow \sum_{i=1}^m \left(B^T \frac{\partial H}{\partial x}(x)\right)_i = 0\\ &\Rightarrow \sum_{i=1}^m \tilde{x}_{c_i}(t) = \sum_{i=1}^m \tilde{x}_{c_i}(0)\\ &\Rightarrow \sum_{i=1}^m \left(-B^T \frac{\partial H}{\partial x}(x(t)) - \tilde{x}_c(t)\right)_i\\ &= \sum_{i=1}^m -\tilde{x}_{c_i}(0), \quad \forall t > 0.\end{aligned} \quad (38)$$

Next, we prove that $\mathbb{1}^T S\left(-B^T \frac{\partial H}{\partial x}(x) - \tilde{x}_c; \bar{x}_c, u^+ + \bar{x}_c\right)$ is bounded from below. Indeed, suppose that $\mathbb{1}^T S\Big(-B^T \frac{\partial H}{\partial x}(x(t)) - \tilde{x}_c(t); \bar{x}_c, u^+ + \bar{x}_c\Big)$ has an increasing sequence $\{t_k\}$ with $t_k \geqslant 0$ converging to $-\infty$, i.e.,

$$\lim_{k\to\infty} \mathbb{1}^T S\left(-B^T \frac{\partial H}{\partial x}(x(t_k)) - \tilde{x}_c(t_k); \bar{x}_c, u^+ + \bar{x}_c\right) = -\infty. \quad (39)$$

Notice that

$$\int_0^{\left(-B^T \frac{\partial H}{\partial x}(x) - \tilde{x}_c\right)_i} \mathrm{sat}(y; \bar{x}_{ci}, u_i^+ + \bar{x}_{ci})\, \mathrm{d}y \quad (40)$$

can have a negative value only when $\left(-B^T \frac{\partial H}{\partial x}(x) - \tilde{x}_c\right)_i < 0$. Therefore we may assume without loss of generality that

$$\begin{aligned}&\left(-B^T \frac{\partial H}{\partial x}(x(t_k)) - \tilde{x}_c(t_k)\right)_i < 0, \quad i \in \mathcal{E}_1 \subset \mathcal{E},\\ &\lim_{k\to\infty} \sum_{i\in\mathcal{E}_1} \left(-B^T \frac{\partial H}{\partial x}(x(t_k)) - \tilde{x}_c(t_k)\right)_i = -\infty.\end{aligned} \quad (41)$$

By Eq. (38), we have

$$\begin{aligned}&\left(-B^T \frac{\partial H}{\partial x}(x(t_k)) - \tilde{x}_c(t_k)\right)_j > 0, \quad j \in \mathcal{E}_2\\ &\lim_{k\to\infty} \sum_{j\in\mathcal{E}_2} \left(-B^T \frac{\partial H}{\partial x}(x(t_k)) - \tilde{x}_c(t_k)\right)_j = +\infty\end{aligned} \quad (42)$$

where $\mathcal{E}_2 = \mathcal{E} \setminus \mathcal{E}_1$. Then Definition 9 implies that $(\bar{x}_c + u^+)_s > \bar{x}_{c_r}$, $\forall s, r = 1, 2, \ldots, m$, which leads to $\lim_{k\to\infty} \mathbb{1}^T S\big(-B^T \frac{\partial H}{\partial x}(x(t_k)) - \tilde{x}_c(t_k); \bar{x}_c, u^+ + \bar{x}_c\big) = +\infty$. This is a contradiction.

Furthermore, since $H(x)$ has a lower bound, then $V(x, \tilde{x}_c)$ is bounded from below for any given initial condition $(x(0), \tilde{x}_c(0))$.

(ii) Notice that $\dot{V} = -\dot{x}^T \dot{x} \leqslant 0$.

Suppose that $x(t)$, $t \geqslant 0$, is not bounded, then there exists a sequence $\{t_k\}$, $t_k \geqslant 0$ such that $\lim_{k\to\infty} \|x(t_k)\| = \infty$. Since $H(x)$ is unbounded, this implies

$$\lim_{k\to\infty} V(x(t_k), \tilde{x}_c(t_k)) = +\infty.$$

This is a contradiction with $\dot{V} \leqslant 0$.

Suppose that $\tilde{x}_c(t)$ is not bounded, then as follows from the proof of (i), there exists a sequence $\{t_k\}$ such that

$$\lim_{k\to\infty} \mathbb{1}^T S\left(-B^T \frac{\partial H}{\partial x}(x(t_k)) - \tilde{x}_c(t_k); \bar{x}_c, u^+ + \bar{x}_c\right) = +\infty$$

which implies $\lim_{k\to\infty} V(x(t_k), \tilde{x}_c(t_k)) = +\infty$. Again this is a contradiction with $\dot{V} \leqslant 0$.

In conclusion, $(x(t), \tilde{x}_c(t))$ is bounded.

(iii) From the dynamics (30) and (ii), it can be shown that $\frac{\mathrm{d}}{\mathrm{d}t}\left(-B^T \frac{\partial H}{\partial x}(x) - \tilde{x}_c\right)$ is bounded. Combining the facts that $V(x, \tilde{x}_c)$ is bounded from below with $\dot{V} \leqslant 0$, we have that $\lim_{t\to\infty} \dot{V}(x(t), \tilde{x}_c(t)) = 0$.

Indeed, suppose $\dot{V}(x(t), \tilde{x}_c(t))$ does not converge to zero. In other words, there exist a real $\delta > 0$ and a sequence $\{t_k\}$, satisfying $\lim_{k\to\infty} t_k = +\infty$, such that $\dot{V}(x(t_k), \tilde{x}_c(t_k)) < -\delta$. Since $\frac{\mathrm{d}}{\mathrm{d}t}\left(-B^T \frac{\partial H}{\partial x}(x) - \tilde{x}_c\right)$ is bounded, then for each $k = 1, 2, \ldots,$ there exist a time interval I_k and an $\epsilon > 0$ such that $|I_k| > \epsilon$, $t_k \in I_k$, and $\forall t \in I_k$, $\dot{V}(x(t), \tilde{x}_c(t)) < -\frac{\delta}{2}$. This implies that

$$\lim_{t\to\infty} V(x(t), \tilde{x}_c(t)) = -\infty,$$

which is a contradiction with (i). In conclusion, $\lim_{t\to\infty} \dot{V}(x(t), \tilde{x}_c(t)) = 0$. $\square$

Proof of Theorem 12. *Sufficiency*: consider now the following function

$$V(x, x_c) = \mathbb{1}^T S\left(-B^T \frac{\partial H}{\partial x}(x) - \tilde{x}_c; \bar{x}_c, u^+ + \bar{x}_c\right) + H(x). \tag{43}$$

Using Lemma 14 and LaSalle's principle, it can be shown that $(x(t), \tilde{x}_c(t))$ converges to the largest invariant set $\mathcal{I}$ contained in $\{(x, \tilde{x}_c) \mid \dot{V} = 0\}$. Similar to the proof of Theorem 6, if a solution $(x(t), \tilde{x}_c(t)) \in \mathcal{I}$, then x is a constant vector, denoted as ν. Furthermore, $\mathcal{I}$ is given as

$$\mathcal{I} = \left\{(\nu, \tilde{x}_c) \mid \tilde{x}_c = B^T \frac{\partial H}{\partial x}(\nu)t + \tilde{x}_c(0), B\mathrm{sat}\left(-B^T \frac{\partial H}{\partial x}(\nu) - B^T \frac{\partial H}{\partial x}(\nu)t - \tilde{x}_c(0); \bar{x}_c, u^+ + \bar{x}_c\right) = 0, \forall t \geq 0\right\}. \tag{44}$$

Suppose now that $B^T \frac{\partial H}{\partial x}(\nu) \neq 0$. Then for t large enough, we have

$$0 = \frac{\partial^T H}{\partial x}(\nu)B\mathrm{sat}\left(-B^T \frac{\partial H}{\partial x}(\nu) - B^T \frac{\partial H}{\partial x}(\nu)t - \tilde{x}_c(0); \bar{x}_c, u^+ + \bar{x}_c\right) = \sum_{i=1}^{m}\left(B^T \frac{\partial H}{\partial x}(\nu)\right)_i c_i, \tag{45}$$

where

$$c_i = \begin{cases} \bar{x}_{ci} & \text{if } \left(B^T \frac{\partial H}{\partial x}(\nu)\right)_i > 0, \\ u_i^+ + \bar{x}_{ci} & \text{if } \left(B^T \frac{\partial H}{\partial x}(\nu)\right)_i < 0. \end{cases} \tag{46}$$

Since the graph is balanced we have $B\mathbb{1}_m = 0$, and thus

$$\sum_{i=1}^{m}\left(B^T \frac{\partial H}{\partial x}(\nu)\right)_i = 0. \tag{47}$$

By the definition of the permission set $\mathcal{P}(0_m, u^+)$, $0 < \bar{x}_{ci} < u_j^+ + \bar{x}_{cj}$ for any $i, j = 1, 2, \ldots, m$, so

$$\sum_{i=1}^{m}\left(B^T \frac{\partial H}{\partial x}(\nu)\right)_i c_i < 0. \tag{48}$$

This yields a contradiction. Hence $B^T \frac{\partial H}{\partial x}(\nu) = 0$ and therefore

$$\mathcal{I} = \left\{(\nu, \tilde{x}_c) \Big| \frac{\partial H}{\partial x}(\nu) = c\mathbb{1}_n, B\mathrm{sat}(-\tilde{x}_c; \bar{x}_c, u^+ + \bar{x}_c) = 0\right\}.$$

Necessity: first, if the graph $\mathcal{G}$ is not strongly connected, then by the same argument as in Theorem 6, it can be easily seen that $\frac{\partial H}{\partial x}(x)$ will not reach consensus.

Now we will show that if the strongly connected network is unbalanced, then there exist a constraint interval $[0_m, u^+]$ and an in/outflow $\bar{d}$ for which there exists $\bar{x}_c \in \mathcal{P}(0_m, u^+)$ such that $E\bar{d} = B\bar{x}_c$ while $\frac{\partial H}{\partial x}(x)$ is not converging to consensus.

For the simplicity of exposition we shall take the set of constraint intervals as $[0_m, \mathbb{1}_m]$.

As in the proof of Lemma 13 we let k be the minimal number of cycles to cover $\mathcal{G}$, and we let $\mathcal{T} = (C_1, \ldots, C_k)$ be a minimal covering set for $\mathcal{G}$. With some abuse of notation

$$BC_i = 0, \quad i = 1, \ldots, k \tag{49}$$

where C_i is the m-dimensional vector whose j-th component is equal to the number of times the j-th edge appears in the cycle C_i.

In the following, we will prove that there exist $B^T \frac{\partial H}{\partial x}(\nu) \neq 0$, $\bar{x}_c \in \mathcal{P}(0_m, \mathbb{1}_m)$, $\tilde{x}_c(0)$ and $\lambda \in \mathbb{R}$, such that

$$\mathrm{sat}\left(-B^T \frac{\partial H}{\partial x}(\nu) - B^T \frac{\partial H}{\partial x}(\nu)t - \tilde{x}_c(0); \bar{x}_c, \mathbb{1}_m + \bar{x}_c\right) = \lambda T, \quad \forall t \geq 0, \tag{50}$$

where ν is the equilibrium value of x as above, and T is the m-dimensional vector whose i-th component is the number of cycles in $\mathcal{T}$ which contain the i-th edge. This implies that the system has an equilibrium $(\nu, \tilde{x}_c)$ which satisfies $\frac{\partial H}{\partial x}(\nu) \notin \mathrm{span}\{\mathbb{1}_n\}$.

Consider as above a minimal covering set $\mathcal{T} = (C_1, \ldots, C_k)$ for $\mathcal{G}$. Let $T_{\max} := \max\{T_i \mid i = 1, 2, \ldots, m\}$, and denote $\mathcal{E}_1 = \{i\text{-th edge} \mid T_i = T_{\max}\}$. Every cycle in $\mathcal{T}$ has at least one non-overlapped edge (see the proof of Lemma 13), and we denote by $\mathcal{E}_2$ the set of all the non-overlapped edges in the cycles in $\mathcal{T}$ which contain at least one edge which is overlapped $l_{\max}$ times.

In the last step, we will make the flows through the edges in $\mathcal{E}_1$ reach the upper bounds of the constraint intervals, and the flows through the edges in $\mathcal{E}_2$ reach their lower bounds. By taking

$$\begin{cases} \frac{\partial H}{\partial x}(\nu)_j < \frac{\partial H}{\partial x}(\nu)_i, & (v_i, v_j) \in \mathcal{E}_1 \\ \frac{\partial H}{\partial x}(\nu)_j > \frac{\partial H}{\partial x}(\nu)_i, & (v_i, v_j) \in \mathcal{E}_2 \\ \frac{\partial H}{\partial x}(\nu)_j = \frac{\partial H}{\partial x}(\nu)_i, & \text{else} \end{cases} \tag{51}$$

for suitable $\bar{x}_c$ and $\tilde{x}_c(0)$, it follows that (50) holds. Indeed, in the set $\mathcal{E}_1 \cup \mathcal{E}_2$, the Eq. (50) takes the form

$$\begin{aligned} &1 + \bar{x}_{cq} = \lambda T_q, \ q\text{-th edge belongs to } \mathcal{E}_1 \\ &\bar{x}_{cp} = \lambda T_p, \ p\text{-th edge belongs to } \mathcal{E}_2 \end{aligned} \tag{52}$$

Now take λ be such that $\frac{1}{l_{\max}} < \lambda < 1$. Then (52) contains $|\mathcal{E}_1| + |\mathcal{E}_2|$ equations and the same number of variables, and has a unique solution $\bar{x}_{cp}, \bar{x}_{cq}$ such that

$$\begin{aligned} &0 < \bar{x}_{cp} < 1 \\ &0 < \bar{x}_{cq} < 1. \end{aligned} \tag{53}$$

Furthermore, pick $\tilde{x}_c(0)$ in the third equation of (51) as

$$\tilde{x}_c(0)_r = -\lambda T_r, \quad r\text{-th edge belongs to } \mathcal{E} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2). \tag{54}$$

Obviously, there exists $0 < \bar{x}_{cr} < 1$ such that

$$\bar{x}_{cr} < -\tilde{x}_c(0)_r < 1 + \bar{x}_{cr} \tag{55}$$

In conclusion, there exists an equilibrium $(\nu, \tilde{x}_c)$ that does not satisfy $B^T \frac{\partial H}{\partial x}(\nu) = 0$, and thus $\frac{\partial H}{\partial x}(x)$ cannot reach the consensus. $\square$

The above constructive proof is illustrated by the following example.

Example 5.1. Consider a directed graph in Fig. 2 with dynamics given by system (30) where $H(x) = \frac{1}{2}\|x\|^2$ and $[u^-, u^+] = [0_7, \mathbb{1}_7]$, that is

$$\begin{aligned} \dot{x} &= B\mathrm{sat}(-B^T x - x_c; 0_7, \mathbb{1}_7) + E\bar{d}, \\ \dot{x}_c &= B^T x. \end{aligned} \tag{56}$$

The purpose of this example is to show that there exist in/outflows $\bar{d}$ satisfying the matching condition for which x does not converge to consensus. By taking $E\bar{d} = B\bar{x}_c$ where $\bar{x}_c = \frac{1}{2}\mathbb{1}_7$, $x(0) = (3, 75, 1, 4)^T$ and $\tilde{x}_c(0) = (1, -1, -1, 1, 1, 1, 1)^T$, the state x in system (56) will converge to ν with $\nu_2 = \nu_3 > \nu_5 > \nu_4 > \nu_1$ and

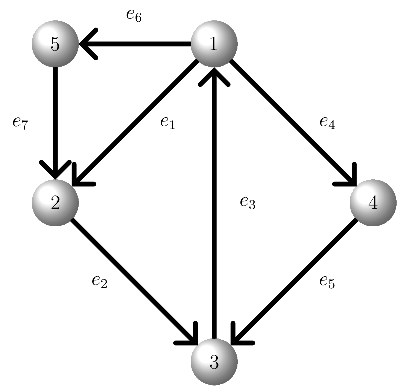


Fig. 2. Network of Example 5.1.

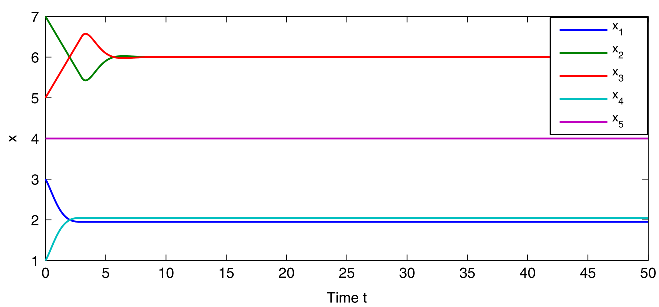


Fig. 3. The time-evolutions $x_1(t), x_2(t), x_3(t), x_4(t), x_5(t)$ of the system (56).

$\nu_4 < \nu_1$ as can be seen from the numerical simulation in Fig. 3. The same result is deduced from the following analysis. In Fig. 2, the smallest number of cycles to cover the whole graph is 3; one option being (e_6, e_7, e_2, e_3), (e_1, e_2, e_3), (e_3, e_4, e_5). So $BT = 0$ where

$$T = (1, 2, 3, 1, 1, 1, 1)^T \tag{57}$$

In this case $\mathcal{E}_1 = \{e_3\}$, $\mathcal{E}_2 = \{e_1, e_4, e_5, e_6, e_7\}$. By setting $\nu_2 = \nu_3 > \nu_5 > \nu_4 > \nu_1$, the flow in e_3 reaches its upper bound, while the flows in e_1, e_4, e_5, e_6, e_7 reach their lower bounds, i.e.,

$$\mathrm{sat}(-B^T\nu - B^T\nu t - \tilde{x}_c(0); \bar{x}_c, \bar{x}_c + \mathbb{1}) = \frac{1}{2}T, \quad \forall t > 0. \tag{58}$$

Thus there exists an equilibrium ν satisfying $B^T\nu \neq 0$.

6. Conclusions

We have discussed a basic model of dynamical distribution networks where the flows through the edges are generated by distributed PI controllers. The resulting system can be naturally modeled as a port-Hamiltonian system, enabling the easy derivation of sufficient and necessary conditions for the convergence of the state variables to load balancing (consensus). The main part of this paper focuses on the case where flow constraints are present. A key ingredient in this analysis is the construction of a C^1 Lyapunov function. We distinguish between the case that the flow constraints corresponding to all the edges allow for bi-directional flow and the case that all the edges only allow for uni-directional flow. For both cases we have derived necessary and sufficient conditions for asymptotic load balancing based on the structure of the graph.

An obvious open problem is the extension of our results to the general case where some of the edges allow for bi-directional flow and others only for uni-directional flow. This is currently under investigation. Many other questions can be addressed in this framework. For example, what is happening if the in/outflows are not assumed to be constant, but are e.g. periodic functions of time; see already [16]. Furthermore, the use of constrained PI-controllers may suggest a fruitful connection to anti-windup control ideas.

Acknowledgments

The work of the first author is supported by the Chinese Science Council (CSC).

The research of the second author leading to these results has received funding from the EU 7th Framework Programme [FP7/2007–2013] under the grant agreement no. 257462 HYCON2 Network of Excellence.

References

[1] A.J. van der Schaft, B.M. Maschke, Port-Hamiltonian systems on graphs, SIAM Journal on Control and Optimization 51 (2) (2013) 906–937.
[2] A.J. van der Schaft, B.M. Maschke, Conservation laws on higher-dimensional networks, in: Proc. 47th IEEE Conf. on Decision and Control, 2008, pp. 799–804.
[3] A.J. van der Schaft, B.M. Maschke, in: P.M.J. Van den Hof, C. Scherer, P.S.C. Heuberger (Eds.), Model-Based Control: Bridging Rigorous Theory and Advanced Technology, Springer, Berlin–Heidelberg, 2009, pp. 31–48. Chapter Conservation laws and lumped system dynamics.
[4] A.J. van der Schaft, B.M. Maschke, Port-Hamiltonian dynamics on graphs: consensus and coordination control algorithms, in: Proc. 2nd IFAC Workshop on Distributed Estimation and Control in Networked Systems, Annecy, France, 2010, pp. 175–178.
[5] M. Bürger, D. Zelazo, F. Allgöwer, Network clustering: a dynamical systems and saddle-point perspective, in: IEEE Conference on Decision and Control, 2011, pp. 7825–7830.
[6] D. Zelazo, M. Mesbahi, Edge agreement: Graph-theoretic performance bounds and passivity analysis, IEEE Transactions on Automatic Control 56 (2011) 544–555. http://dx.doi.org/10.1109/TAC.2010.2056730.
[7] A.J. van der Schaft, J. Wei, A Hamiltonian perspective on the control of dynamical distribution networks, in: 4th IFAC Workshop on Lagrangian and Hamiltonian Methods for Non Linear Control, 2012, pp. 24–29.
[8] B. Bollobas, Modern Graph Theory, in: Graduate Texts in Mathematics, vol. 184, Springer, New York, 1998.
[9] A.J van der Schaft, B.M. Maschke, The Hamiltonian formulation of energy conserving physical systems with external ports, Archiv für Elektronik und Übertragungstechnik 49 (1995) 362–371.
[10] A.J. van der Schaft, L_2-Gain and Passivity Techniques in Nonlinear Control, second ed., in: Lect. Notes in Control and Information Sciences, vol. 218, Springer-Verlag, Berlin, 1996, Springer, London, 2000.
[11] J.G. Romero, A. Donaire, R. Ortega, Robustifying energy shaping control of mechanical systems, in: 51st IEEE Conference on Decision and Control, 2012, pp. 4424–4429.
[12] R. Ortega, A.J. van der Schaft, F. Castanos, A. Astolfi, Control by interconnection and standard passivity-based control of port-Hamiltonian systems, IEEE Transactions on Automatic Control 53 (11) (2008) 2527–2542.
[13] B. Jayawardhana, R. Ortega, E. García-Canseco, F. Castaños, Passivity of nonlinear incremental systems: application to PI stabilization of nonlinear RLC circuits, Systems & Control Letters 56 (2007) 618–622.
[14] F. Blanchini, S. Miani, W. Ukovich, Control of production–distribution systems with unknown inputs and system failures, IEEE Transactions on Automatic Control 45 (6) (2000) 1072–1081.
[15] W. Ren, On consensus algorithms for double-integrator dynamics, IEEE Transactions on Automatic Control 53 (6) (2008).
[16] C.D. Persis, Balancing time-varying demand-supply in distribution networks: an internal model approach, in: ECC 2013 Zurich Proceedings.